ON LEAST ENERGY SOLUTIONS TO A SEMILINEAR ELLIPTIC EQUATION IN A STRIP

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Dedicated to Professor L. Nirenberg on the occasion of his 85th birthday,
with deep admiration

ABSTRACT. We consider the following semilinear elliptic equation on a strip:

\[
\begin{aligned}
\Delta u - u + u^p &= 0 \text{ in } \mathbb{R}^{N-1} \times (0, L), \\
u > 0, \frac{\partial u}{\partial v} &= 0 \text{ on } \partial(\mathbb{R}^{N-1} \times (0, L))
\end{aligned}
\]

where \(1 < p \leq \frac{N+2}{N-2}\). When \(1 < p < \frac{N+2}{N-2}\), it is shown that there exists a unique \(L_* > 0\) such that for \(L \leq L_*\), the least energy solution is trivial, i.e., doesn’t depend on \(x_N\), and for \(L > L_*\), the least energy solution is nontrivial. When \(N \geq 4, p = \frac{N+2}{N-2}\), it is shown that there are two numbers \(L_* < L_{**}\) such that the least energy solution is trivial when \(L \leq L_*\), the least energy solution is nontrivial when \(L \in (L_*, L_{**}]\), and the least energy solution does not exist when \(L > L_{**}\). A connection with Delaunay surfaces in CMC theory is also made.

1. INTRODUCTION

In this paper, we consider the following semilinear elliptic equation on a strip

\[
\begin{aligned}
\Delta u - u + u^p &= 0 \text{ in } \mathbb{R}^{N-1} \times (0, L), \\
\frac{\partial u}{\partial v} &= 0 \text{ on } \partial(\mathbb{R}^{N-1} \times (0, L)), \\
u > 0, u \in H^1(\mathbb{R}^{N-1} \times (0, L)).
\end{aligned}
\]

(1.1)

Here we assume that

\[N \geq 2, 1 < p \leq \frac{N+2}{N-2} \text{ if } N \geq 3, \text{ and } 1 < p < +\infty \text{ if } N = 2\]

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and \( \nu \) is the outer normal derivative.

The motivation of this study stems from the work of Dancer on new solutions to the following simple superlinear problem

\[
\Delta u - u + u^p = 0 \quad \text{in} \quad \mathbb{R}^N, \quad u > 0, \quad p > 1.
\]

If \( u(x) \to 0 \) as \( |x| \to +\infty \), then the classical work of Gidas-Ni-Nirenberg \cite{12} shows that \( u \) must be \textit{radially symmetric} with respect to one point and thus (1.2) is reduced to an ODE. On the other hand, there are less known results on solutions to (1.2) which do not decay in all directions. Dancer \cite{6} first constructed solutions to (1.2) that are periodic in one direction and decays in all the other directions, via local bifurcation arguments. They form a one-parameter family of solutions which are periodic in the \( z \) variable and originate from the decaying solutions of (1.2) in \( \mathbb{R}^{N-1} \). We briefly outline Dancer’s idea: Let \( T > 0 \) be the period and consider

\[
\begin{aligned}
&\Delta u - u + u^p = 0 \quad \text{in} \quad \mathbb{R}^N, \quad u > 0, \\
&u(x', x_N + T) = u(x', x_N), \quad u(x', x_N) \to 0 \quad \text{as} \quad |x'| \to +\infty
\end{aligned}
\]

where we denote \( x' = (x_1, \ldots, x_{N-1}) \). Dancer then used \( T \) as the bifurcation parameter and found a critical value \( T_1 \) such that for \( T = T_1 \), the linearized problem at the lower dimensional decaying solutions has an eigenvalue zero with eigenfunctions decaying in \( x' \). Then using the Crandall-Rabinowitz bifurcation theory, near \( T_1 \), a new solution (different from lower dimensional solution) bifurcates.

In \cite{10}, these periodic solutions are called \textit{Dancer’s solutions} and they are the building blocks for more complicated “2k-ends” solutions. In \cite{22}, Dancer’s solutions are also used to build \textit{three ends} solutions to the problem in entire space. In fact, geometrically, Dancer’s solutions corresponds to the so-called Delaunay solution in CMC theory \cite{8}. We will comment on this later. Therefore it becomes natural to study the solution structure of (1.1).

Problem (1.1) also arises naturally in the study of some nonlinear elliptic equations in an expanding annuli:

\[
\begin{aligned}
&\Delta u - u + u^p = 0 \quad \text{in} \quad B_{R+L} \setminus B_R, \\
&u > 0, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial (B_{R+L} \setminus B_R),
\end{aligned}
\]

where \( R \to +\infty \) and \( L \) is fixed. The limiting equation of (1.4) as \( R \to +\infty \) becomes (1.1).

We note that the corresponding expanding annuli Dirichlet problem

\[
\begin{aligned}
&\Delta u - u + u^p = 0 \quad \text{in} \quad B_{R+L} \setminus B_R, \\
&u > 0, \quad u = 0 \quad \text{on} \quad \partial (B_{R+L} \setminus B_R),
\end{aligned}
\]

has been studied by many authors, see \cite{4}, \cite{5}, \cite{7}, \cite{17}, \cite{18}, \cite{19}, \cite{23} and the references therein. We are not aware of any study on (1.4). Note that (1.4) is different from (1.5). Indeed, (1.4) admits solutions that are nonzero and
lower dimensional, i.e. don’t depend on $x_N$-direction. The issue, therefore, is to understand when and how solutions that are not lower dimensional exist.

By suitable scaling (1.1) becomes

\[
\begin{aligned}
\Delta u - L^2 u + u^p &= 0 \text{ in } \Sigma := \mathbb{R}^{N-1} \times (0, 1), \\
\frac{\partial u}{\partial n} &= 0 \text{ on } \partial \Sigma, \quad u > 0, \; u \in H^1(\Sigma).
\end{aligned}
\]

Here $L$ is the parameter. In this paper, we consider the existence or nonexistence as well as nature of least energy solutions. More precisely, let

\[
(1.7) \quad c(L) := \inf_{u \in H^1(\Sigma), u \not\equiv 0} \frac{\int_{\Sigma} (|\nabla u|^2 + L^2 u^2)}{(\int_{\Sigma} u^{p+1})^{\frac{2}{p+1}}}.
\]

Our main concerns are:

Q1: Is $c(L)$ attained?

Q2: Is $c(L)$ attained by a nontrivial solution?

Q3: Is the least energy solution nondegenerate?

Here, a trivial solution is understood to mean that the solution does not depend on $x_N$. Note that for $p < \frac{N+2}{N-2}$ if $N \geq 4$ and $1 < p < +\infty$ if $N = 2, 3$, such trivial solutions of (1.6) always exist. Indeed, a ground state solution ([2]-[3]) in $\mathbb{R}^{N-1}$ yields such a “trivial solution”.

Our main theorem is

**Theorem 1.1.** (1) If $p < \frac{N+2}{N-2}$ when $N \geq 3$ and $p < +\infty$ when $N = 2$, then there exists a unique $L_*$ such that for $L \leq L_*$, $c(L)$ is attained by a trivial solution and for $L > L_*$, $c(L)$ is attained by a nontrivial solution.

(2) There exist $L_2 \geq L_*$ such that the least energy solution is unique and nondegenerate for any $L \geq L_2$.

(3) If $N \geq 4, p = \frac{N+2}{N-2}$, then there exists two positive constants $L_* < L_{**}$ such that for $L \leq L_*$, $c(L)$ is attained by a trivial solution; for $L \in [L_*, L_{**}]$, $c(L)$ is attained by a nontrivial solution; for $L > L_{**}$, $c(L)$ is not attained.

**Remark:** The number $L^*$ can be computed as follows: Let $w_0$ be the unique ground state solution in $\mathbb{R}^{N-1}$

\[
(1.8) \quad \Delta w_0 - w_0 + w_0^p = 0, \quad w_0 = w_0(|x'|) > 0, \; w_0 \in H^1(\mathbb{R}^{N-1}).
\]

(See [2], [15].) Let $\lambda_1$ be the unique principal eigenvalue of

\[
(1.9) \quad \Delta \phi - \phi + p w_0^{p-1} \phi = \lambda_1 \phi, \; \phi \in H^1(\mathbb{R}^{N-1}).
\]

Then we have

\[
(1.10) \quad L^* = \frac{\pi}{\sqrt{\lambda_1}}
\]

When $N = 2$, we can compute explicitly (see [9])

\[
(1.11) \quad \lambda_1 = \frac{(p-1)(p+3)}{4}.
\]
In fact, we can say more about the properties of the minimizers and the asymptotic behaviors of \( c(L) \) as \( L \to 0 \) or \( L \to +\infty \). The asymptotic behavior of the least energy solution when \( L \to L_* \) is given in the appendix.

Even though Theorem 1.1 is a purely PDE result, this result has a striking analogy in the theory of constant mean curvature (CMC) surface in \( \mathbb{R}^3 \).

CMC surfaces in \( \mathbb{R}^3 \) are equilibria for the area functional subjected to an enclosed volume constraint. It arises in many physical and variational problems. Over the past two decades a great deal of progress was achieved in understanding complete CMC surfaces and their moduli spaces. Spheres (zero end) and round cylinders are the first examples of CMC surfaces. (See Alexandrov’s [1].) Properly embedded CMC surfaces with nonzero mean curvature were classified by Delaunay [8]. These are CMC rotation surfaces, called unduloids (having genus zero and two ends). These surfaces are derived from two 1-parameter families: one of the family being unduloids with neck radius \( \tau \in (0, \frac{1}{2}] \) and the other being a family of non-embedded surface called nodoids that can be parameterized by the neck radius \( \tau \in (0, \infty) \).

In very much an analogous way, our solutions in Theorem 1.1 are parameterized by the length \( L \). When \( L \to +\infty \), these solutions become spikes at the center and correspond to the Delaunay surface that are obtained when \( \tau \to 0 \). On the other hand, when \( L \to L_* \) our solution corresponds to Delaunay solution when \( \tau \to \frac{1}{2} \). One good way to think of this analogy is the level sets of \( u \). (See [10] for more explanations.) In [10], del Pino-Kowalczyk-Pacard-Wei used the least energy solution near \( L^* \) and Toda systems to build more complicated even-ended solutions of (1.2) in \( \mathbb{R}^2 \), while in [22], Malchiodi used the least energy solution near \( +\infty \) to build \( Y \)-shaped solutions.

We conjecture that the least energy solution form a continuous family as \( L \) goes from \( L_* \) to \( +\infty \).

After the paper was completed, we learned from Prof. M. Esteban that problem (1.6) is also related to the study of Caffarelli-Kohn-Nirenberg inequality and it is studied in the work of Dolbeault, Esteban, Loss and Tarantello [11].

This paper is organized as follows: we prove (1), (2) and (3) of Theorem 1.1 in Sections 2,3 and 4 respectively. In Appendix A, we prove some technical estimates used in Section 4 while in Appendix B we study the asymptotic behavior of least energy solutions when \( L \to L_* \).

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2. The subcritical case: Proof of (1) of Theorem 1.1

In this section, we study (1.6) for the subcritical case, i.e., $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$ and $1 < p < +\infty$ when $N = 2$. We begin with

**Lemma 2.1.** For any $L_0 > 0$, there exists a constant $C$, independent of $L \leq L_0$, such that for any solution of (1.6) we have

\begin{equation}
    u \leq C.
\end{equation}

*Proof.* This follows from standard blowing-up argument. For the sake of completeness, we include a short proof here. Suppose (2.1) were not true. Then, there would exist a sequence of functions $u_i$ and $L_i \leq L_0$ such that $M_i := \sup_{x \in \Sigma} u_i(x) \rightarrow +\infty$. Without loss of generality, we may assume that $M_i = u_i(x_i), x_i = (0', x_{i,N})$. Indeed, for fixed $i$, there exists a sequence of points $x_i^k$ such that $u_i(x_i^k) \rightarrow \sup_{x \in \Sigma} u_i$ as $k \rightarrow +\infty$. Then let $x_i^k = ((x_i^k)', x_{i,N}^k)$ and define $u_i^k(x) = u_i(x + (x_i^k)' , x_N)$. Then, using standard elliptic estimates, one can strike out a sequence $u_i^k$ converging to $u_i^\infty$ as $k \rightarrow +\infty$. We may then replace $u_i$ by $u_i^\infty$ (which we call $u_i$ again). Then, $u_i$ is a solution of (1.6) and there exists $x_i = (0', x_N')$ such that $u_i(x_i) = \max_{\Sigma} u := M_i$.

Then we perform a classical blow up analysis. Set

\begin{equation}
    \epsilon_i = M_i^{-\frac{p-1}{2}}, v_i(y) = \epsilon_i^{\frac{2}{p-1}} u_i(x_i + \epsilon_i y).
\end{equation}

Then it is easy to see that $v_i(y)$ satisfies

$$
\Delta v_i - L_i^2 \epsilon_i^2 v_i + v_i^p = 0 \text{ in } \Sigma_i = \mathbb{R}^{N-1} \times \left(-\frac{x_{i,N}}{\epsilon_i}, \frac{L_i - x_{i,N}}{\epsilon_i}\right).
$$

Now we extend $v_i$ to $\mathbb{R}^N$ be periodic extension. We still denote the periodic extension as $v_i$. Then, up to extraction of a subsequence, $v_i(y) \rightarrow v_0(y)$ in $C^2_{\text{loc}}(\mathbb{R}^N)$, and $v_0$ satisfies the equation

$$
\Delta v_0 + v_0^p = 0 \text{ in } \mathbb{R}^N, v_0(0) = 1.
$$

However, this is clearly impossible by the result of Gidas-Spruck [13].

As a corollary, we have

**Corollary 2.2.** There exists a $L_* > 0$ such for $L < L_*$, $c(L_i)$ is achieved only by trivial solutions.

*Proof.* Certainly by Sobolev embedding theorem and Steiner’s symmetrization, a minimizer which is symmetric in $x'$ to $c(L)$ exists. We call it $u_L$. Now consider the function $\phi = \frac{\partial u_L}{\partial x_N}$. Then $\phi$ satisfies

\begin{equation}
    \Delta \phi - L^2 \phi + p u_L^{p-1} \phi = 0 \text{ in } \Sigma, \phi = 0 \text{ on } \partial \Sigma.
\end{equation}

As $L \rightarrow 0$, since $u_L$ is uniformly bounded by Lemma 2.1 and by the same argument as in the proof of Lemma 2.1, we conclude that $\sup_{x \in \Sigma} u_L \rightarrow 0.$
On the other hand, since \( \phi \in H^1_0(\Sigma) \), by Poincare’s inequality
\[
(2.4) \quad C \int_\Sigma |\phi|^2 \leq \int_\Sigma |\nabla \phi|^2
\]
for some positive constant \( C \).

Multiplying (2.3) by \( \phi \), we obtain using (2.4)
\[
(2.5) \quad p \int_\Sigma u_L^{p-1} \phi^2 = \int_\Sigma |\nabla \phi|^2 + L^2 \int_\Sigma \phi^2 \geq C \int_\Sigma \phi^2.
\]
This is impossible if \( \phi \not= 0 \), for \( L \) small enough since \( \sup_{x \in \Sigma} u_L \to 0 \). Therefore for \( L \) small enough, \( \frac{\partial u_L}{\partial x_N} \equiv 0 \) and \( u_L \) is independent of \( x_N \). \( \square \)

Let us denote by \( c^*(L) \) the energy level of the trivial solutions, i.e., solutions depending on \( x' \) only. By a simple computation, we see that
\[
(2.6) \quad c^*(L) = \gamma_0 L^{2-\frac{N-1}{p+1}}.
\]
where \( \gamma_0 \) is generic constant (independent of \( L \)). Certainly \( c(L) \leq c^*(L) \).

**Lemma 2.3.** As \( L \to +\infty \), \( L^{-\frac{2}{p+1}} u_L(L^{-1}y) \to w(y) \) where \( w(y) \) is the unique solution of
\[
(2.7) \quad \left\{ \begin{array}{l}
\Delta w - w + w^p = 0 \text{ in } \mathbb{R}^N, \\
w > 0 \text{ in } \mathbb{R}^N, w(0) = \max_{y \in \mathbb{R}^N} w(y), w(y) \to 0 \text{ as } |y| \to +\infty
\end{array} \right.
\]
In fact, \( u_L \) has only one local maximum point \( P_L \) on \( \partial \Sigma \).

**Proof.** By Schwarz spherical rearrangement with respect to \( x' \) and Steiner monotone increasing rearrangement in \( x_N \), after a shift of the origin and a change \( x_N \) to \(-x_N \) if needed, we see that \( u_L \) is radially symmetric in \( x' \) and monotone increasing in \( x_N \). Therefore, there exists a unique point \( P_0 = (0',1) \) where \( u_L \) achieves a maximum. Let \( v_L(y) = L^{-\frac{2}{p+1}} u_L(P_0 + L^{-1}y) \). Then \( v_L \) satisfies
\[
\Delta v_L - v_L + v_L^p = 0 \text{ in } \Sigma_L, v_L \in H^1(\Sigma_L)
\]
where \( \Sigma_L = \mathbb{R}^{N-1} \times (-L,0) \). Simple computations show that
\[
(2.8) \quad c(L) = \left( \int_{\Sigma_L} u_L^{p+1} \right)^{\frac{p-1}{p+1}} = L^{2-\frac{N(p-1)}{p+1}} \left( \int_{\Sigma_L} v_L^{p+1} \right)^{\frac{p-1}{p+1}}.
\]
Since \( w \) decays exponentially, we can use a cut-off of \( w \) to be a test function and derive that
\[
(2.9) \quad c(L) \leq \left( \frac{1}{2} \int_{\mathbb{R}^N} w^{p+1} + O(L^{-1}) \right)^{\frac{p+1}{p+1}} L^{2-\frac{N(p-1)}{p+1}}
\]
for \( L \gg 1 \). We conclude that \( \int_{\Sigma_L} v_L^{p+1} \) is bounded and hence the \( H^1(\Sigma_L) \) norm of \( v_L \) is bounded. Consequently, standard arguments show that as \( L \to +\infty \), \( v_L(y) \to w(y) \). \( \square \)
The following lemma gives part of (1) of Theorem 1.1.

**Lemma 2.4.** Let \( L_0 \) be such that \( c(L_0) < c^*(L_0) \). Then for any \( L > L_0 \), it holds that \( c(L) < c^*(L) \).

**Proof.** Since \( c(L_0) < c^*(L_0) \), \( c(L_0) \) is attained by a nontrivial solution, which we call \( v_0(x', x_N) \). Thus, \( v_0 \) satisfies

\[
\Delta v_0 - L_0^2 v_0 + v_0^p = 0 \text{ in } \Sigma.
\]

Note that since \( v_0 \) is nontrivial, we have

\[
\int_{\Sigma} v_0^2 > 0.
\]

Now let us consider the following transformation:

\[
u(x', x_N) = \lambda^{\frac{p-1}{p+1}} v_0(\lambda x', x_N), \text{ where } \lambda = \frac{L}{L_0}.
\]

A simple computation shows that \( u(x) \) satisfies

\[
\Delta u - L^2 u + u^p + (\lambda^2 - 1)u_{x_N x_N} = 0.
\]

Hence

\[
\int_{\Sigma} (|\nabla u|^2 + L^2 u^2) = \int_{\Sigma} u^{p+1} - (\lambda^2 - 1) \int_{\Sigma} |u_{x_N}|^2.
\]

Now, since \( \lambda > 1 \), we derive the following sequence of inequalities

\[
c(L) \leq \frac{\int_{\Sigma} (|\nabla u|^2 + L^2 u^2)}{(\int_{\Sigma} u^{p+1})^{\frac{2}{p+1}}} < \left( \int_{\Sigma} u^{p+1} \right)^{\frac{p-1}{p+1}} = \lambda^{2 - \frac{p(N-1)}{p+1}} \left( \int_{\Sigma} v_0^{p+1} \right)^{\frac{p-1}{p+1}} < \lambda^{2 - \frac{p(N-1)(p-1)}{p+1}} c^*(L_0) = c^*(L).
\]

The following inequality may be of independent interest.

**Lemma 2.5.** Let \( u_L \) be a least energy solution to \( c(L) \). Then it holds

\[
\int_{\Sigma} \left[ (|\nabla \varphi|^2 + L^2 \varphi^2) - p u_L^{p-1} \varphi^2 \right] + (p-1) \frac{(\int_{\Sigma} u_L^p \varphi)^2}{\int_{\Sigma} u_L^{p+1}} \geq 0, \forall \varphi \in H^1(\Sigma).
\]

**Proof.** This follows from the variational characterizations of \( u_L \). In fact, let

\[
Q[u] = \frac{\int_{\Sigma} (|\nabla u|^2 + L^2 u^2)}{(\int_{\Sigma} u^{p+1})^{\frac{2}{p+1}}}
\]
and \( \rho(t) = Q[u_L + t\varphi] \) for any \( \varphi \in H^1(\Sigma) \). Since \( \rho(t) \geq \rho(0) \) for any \( t \), \( \rho'(0) = 0 \), \( \rho''(0) \geq 0 \). Note that

\[
\int_{\Sigma} (|\nabla (u_L + t\varphi)|^2 + L^2(u_L + t\varphi)^2) - 2t \int_{\Sigma} (\nabla u_L \nabla \varphi + L^2 u_L \varphi) + t^2 \int_{\Sigma} (|\nabla \varphi|^2 + L^2 \varphi^2)
\]

\[
= \int_{\Sigma} u_L^{p+1} + 2t \int_{\Sigma} u_L^p \varphi + t^2 \int_{\Sigma} (|\nabla \varphi|^2 + L^2 \varphi^2)
\]

\[
- \int_{\Sigma} (u_L + t\varphi)^{p+1} = \int_{\Sigma} u_L^{p+1} + (p+1)t \int_{\Sigma} u_L^p \varphi + \frac{(p+1)p}{2} \int_{\Sigma} u_L^{p-1} \varphi^2 + O(t^2)
\]

Then that \( \rho''(0) \geq 0 \) is equivalent to

\[
(2.16) \quad \int_{\Sigma} \left[ (|\nabla \varphi|^2 + L^2 \varphi^2) - pu_L^{p-1} \varphi^2 \right] + (p-1) \frac{(\int_{\Sigma} u_L^p \varphi)^2}{\int_{\Sigma} u_L^{p+1}} \geq 0.
\]

The next result is a corollary of inequality (2.15).

**Lemma 2.6.** Let \( u_L \) be a least energy solution of \( c(L) \) and \( \lambda_2(u_L) \) be the second eigenvalue of

\[
(2.17) \quad \Delta \phi - L^2 \phi + pu_L^{p-1} \phi + \lambda \phi = 0 \text{ in } \Sigma, \quad \frac{\partial \phi}{\partial \nu} = 0 \text{ on } \partial \Sigma.
\]

Then, necessarily

\[
(2.18) \quad \lambda_2(u_L) \geq 0.
\]

**Proof.** Recall that by the Courant-Fisher-Weyl formula, one has

\[
(2.19) \quad \lambda_2 = \max_{\dim(V) = 1, V \in H^1(\Sigma) \varphi \perp V} \inf \int_{\Sigma} \left[ (|\nabla \varphi|^2 + L^2 \varphi^2) - pu_L^{p-1} \varphi^2 \right].
\]

Then by choosing \( V = \text{span}\{u_L^p\} \) in (2.19) and using (2.15), we derive that \( \lambda_2 \geq 0 \). □

**Completion of proof of (1) of Theorem 1.1:** Let

\[
L_\ast = \sup \{ L | c(l) = c^*(l) \text{ for } l \in (0, L) \}.
\]

By Corollary 2.2, we see that \( 0 < L_\ast \). Now from Lemma 2.3, it follows that \( L_\ast < +\infty \). Indeed, for \( L \) large, we have by (2.9) and Lemma 2.3, \( c(L) \sim L^{2-\frac{N(p-1)}{p+1}} \), while \( c^*(L) \sim L^{2-\frac{(N-1)(p-1)}{p+1}} \). Certainly, for \( l \leq L_\ast \), \( c(l) = c^*(l) \) and \( u_l \) is trivial. By Lemma 2.4, \( c(L) < c^*(L) \) for \( L > L_\ast \).
We now claim that $L^* = \frac{x}{\sqrt{\lambda}}$. In fact, by separation of variables, the second eigenvalue of the following eigenvalue problem

\begin{equation}
\Delta \phi - L^2 \phi + pw_0^{p-1}\phi + \lambda \phi = 0 \text{ in } \Sigma, \quad \frac{\partial \phi}{\partial \nu} = 0 \text{ on } \partial \Sigma
\end{equation}

is

\begin{equation}
\lambda_2 = \pi^2 - L^2 \lambda_1 \geq 0.
\end{equation}

By Lemma 2.6, $L^* \leq \frac{x}{\sqrt{\lambda}}$. On the other hand, if $L^* > \frac{x}{\sqrt{\lambda}}$, then by Lemma 2.4, $w_0$ is a minimizer of $c_L$ when $L$ is close to $\frac{x}{\sqrt{\lambda}}$. But it is easy to see that $w_0$ loses its stability exactly at $L = \frac{x}{\sqrt{\lambda}}$ by (2.21).

3. The Subcritical Case: The Proof of (2) of Theorem 1.1

In this section, we prove the uniqueness and nondegeneracy of the least energy solutions for $L \to +\infty$.

First we recall

**Lemma 3.1.** Let $w$ be the least energy solution of

\begin{equation}
\Delta w - w + w^p = 0, \quad w > 0, \quad w \in H^1(\mathbb{R}^N).
\end{equation}

Then $w$ is nondegenerate, i.e.,

\begin{equation}
\text{Ker } (\Delta - 1 + pw^{p-1}) = \text{span } \{ \frac{\partial w}{\partial y_1}, ..., \frac{\partial w}{\partial y_N} \}
\end{equation}

**Proof.** The result is well-known. By the classical result of Gidas-Ni-Nirenberg [12], $w$ is radially symmetric. The nondegeneracy follows from the uniqueness result of Kwong [15]. See Lemma A.3 of [25]. Here we include a short and self-contained new proof of nondegeneracy using only property of least energy. This part is of independent interest, but it is restricted to the power nonlinearity.

Let $w = w(r)$ be a radial least energy solution of (2.7). By the same proof as in Lemma 2.6, $\lambda_{2,r}(w) \geq 0$, where $\lambda_{2,r}$ denotes the second eigenvalue in the radial class. It remains to show that $\lambda_{2,r}(w) > 0$. Suppose $\lambda_{2,r} = 0$ and let $\phi(r)$ be the corresponding eigenfunction, i.e.

\begin{equation}
\Delta \phi - \phi + pw^{p-1}\phi = 0, \quad \phi = \phi(r) \in H^1(\mathbb{R}^N).
\end{equation}

Then the characterization of the second eigenfunction implies that $\phi$ changes sign once. So we may assume that $\phi < 0$ for $r < r_0$ and $\phi > 0$ for $r > r_0$. Now as in Kwong-Zhang [16] we consider the function

\begin{equation}
\eta(r) = rw' - \beta w.
\end{equation}

Then $\eta$ satisfies

\begin{equation}
\Delta \eta - \eta + pw^{p-1}\eta = 2w - (2 + \beta(p - 1))w^p.
\end{equation}
We choose $\beta$ such that $1 = (1 + \frac{\beta(p-1)}{2})w^{p-1}(r_0)$, hence $2w - (2 + \beta(p-1))w^p < 0$ for $r < r_0$ and $2w - (2 + \beta(p-1))w^p > 0$ for $r > r_0$.

Multiplying (3.3) by $\eta$ and (3.5) by $\eta$, we arrive at

\begin{equation}
(3.6) \quad \int_{\mathbb{R}^N} \phi(2w - (2 + \beta(p-1))w^p) = 0
\end{equation}

which is impossible by the property of $\phi$. Thus $\phi \equiv 0$ and this completes the proof. \hfill \Box

Let us now prove the nondegeneracy of the least energy solution when $L$ is large. By the rescaling $w_L = L^{-\frac{2}{p-1}}u_L((0, 1) + L^{-1}y)$, it is enough to show that the only solutions to

\begin{equation}
(3.7) \quad \Delta \phi - \phi + pw^{p-1}\phi = 0, \phi \in H^1(\Sigma_L)
\end{equation}

are $\frac{\partial w_L}{\partial y_j}$, $j = 1, ..., N - 1$. Here $\Sigma_L = \mathbb{R}^{N-1} \times (-L, 0)$. We may assume that

\begin{equation}
(3.8) \quad \int_{\Sigma_L} \phi \frac{\partial w_L}{\partial y_j} = 0, j = 1, ..., N - 1.
\end{equation}

Suppose there is a nonzero solution $\phi$ to (3.7)-(3.8). We may assume that $\|\phi\|_{L^\infty} = 1$. Since $w_L \rightarrow w(y)$ as $L \rightarrow \infty$, we conclude that $pw^{p-1} < \frac{1}{2}$ for $y \in B_R(0) \cap \Sigma_L$. Thus $|\phi(y)| \leq \max_{y \in B_R(0)} |\phi|e^{-\frac{1}{\sqrt{2}}(|y|-R)}$. So the maximum point of $|\phi|$ must occur in $B_{2R}(0)$. Letting $L \rightarrow +\infty$, we have that $\phi \rightarrow \phi_\infty$ which satisfies

\begin{equation}
(3.9) \quad \Delta \phi_\infty - \phi_\infty + pw^{p-1}\phi_\infty = 0, \int_{\mathbb{R}^N} \phi_\infty \frac{\partial w}{\partial y_j} = 0, j = 1, ..., N - 1.
\end{equation}

By Lemma 3.1, $\phi_\infty = c\frac{\partial w}{\partial y_N} = c\frac{u'}{r}y_N$. We have seen that $\phi_\infty$ attains its maximum at some finite point. This is impossible since $\frac{\partial^2 w}{\partial y_N^2} \neq 0$.

The proof of uniqueness of least energy solution when $L$ is large is similar to that of nondegeneracy. In fact, suppose that there are two least energy solutions $u_L$ and $u'_L$ to (1.6). We may assume that both $u_L$ and $u'_L$ attain their maximum at $(0, 1)$. Suppose that $u_L \neq u'_L$. Then letting $w_L(y) := L^{-\frac{2}{p-1}}u_L((0, 1) + L^{-1}y)$, $w'_L(y) := L^{-\frac{2}{p-1}}u'_L((0, 1) + L^{-1}y)$, $\phi_L(y) := w_L(y) - w'_L(y)$, we see that $\phi_L$ satisfies

\begin{equation}
(3.10) \quad \Delta \phi - \phi + V(y)\phi = 0, \phi \in H^1(\Sigma_L)
\end{equation}

where $V(y) = \frac{w'_L - w}{w_L - w_L}$. Since $V(y) \rightarrow pw^{p-1}(y)$ as $L \rightarrow +\infty$ and $\nabla \phi_L(0) = 0$, the rest of the proof is exactly the same as before. We omit the details.
4. The critical exponent Case: the proof of (3) of Theorem 1.1

In this section, we assume that \( p = \frac{N+2}{N-2} \). We consider two cases: \( L \) is small and \( L \) is large.

4.1. Critical exponent case I: \( L \) small. It is well-known (see [13]) that the solutions to the following problem

\[
\Delta u + u^{\frac{N+2}{N-2}} = 0 \quad \text{in} \quad \mathbb{R}^N, \quad u > 0
\]

are given by

\[
U_{c,a} = c_N \left( \frac{\epsilon}{\epsilon^2 + |x-a|^2} \right)^{\frac{N-2}{2}}
\]

for some \( \epsilon > 0 \) and \( a \in \mathbb{R}^N \).

Let

\[
S = \frac{\int_{\mathbb{R}^N} |\nabla U_{1,0}|^2}{\left( \int_{\mathbb{R}^N} U_{1,0}^{\frac{2N}{N-2}} \right)^\frac{N-2}{N}}, \quad S_{\frac{1}{2}} = \left( \frac{1}{2} \right)^\frac{N}{2} S.
\]

We have the following lemma, whose proof follows from classical “concentration-compactness” principle of P.L. Lions [20], [21].

**Lemma 4.1.** Let \( p = \frac{N+2}{N-2} \). If

\[
c(L) < S_{\frac{1}{2}}
\]

then \( c(L) \) is attained.

As a Corollary, we have

**Corollary 4.2.** Let \( N \geq 4 \). Then for \( L \) sufficiently small, \( c(L) \) is attained.

**Proof.** We just need to verify (4.4) for \( L \) small. Now we compute

\[
\int_{\Sigma} U_{c,0}^{\frac{2N}{N-2}}
\]

\[
= c_N^{\frac{2N}{N-2}} \int_{\Sigma} \left( \frac{\epsilon}{\epsilon + |x|^2} \right)^N dx
\]

\[
= c_N^{\frac{2N}{N-2}} \int_0^{\frac{1}{N-1}} \left( \int_{\mathbb{R}^{N-1}} \frac{1}{(1 + t^2 + |y|^2)^N} dy \right) dt
\]

\[
= c_N^{\frac{2N}{N-2}} \int_0^{\frac{1}{N-1}} (1 + t^2)^{-\frac{N+1}{2}} dt \int_{\mathbb{R}^{N-1}} \frac{1}{(1 + |y|^2)^N} dy
\]

\[
= c_N^{\frac{2N}{N-2}} \left[ \int_{\mathbb{R}^N} \frac{1}{(1 + |y|^2)^N} dy - \frac{\epsilon^N}{N} \int_{\mathbb{R}^{N-1}} \frac{1}{(1 + |y|^2)^N} dy' + o(\epsilon^N) \right].
\]
Similarly
\begin{equation}
\int_{\Sigma} |\nabla U_{\epsilon,0}|^2 = c_N^2 (N - 2) \int_{\mathbb{R}^N_y} |\nabla U_{1,0}|^2 dy
\end{equation}

\[-c_N^2 \epsilon^{N-2} \int_{\mathbb{R}^{N-1}} \frac{1}{(1 + |y'|^2)^{N-1}} dy' + c_N^2 \epsilon^N (N - 2) \int_{\mathbb{R}^{N-1}} \frac{1}{(1 + |y'|^2)^N} dy' + O(\epsilon^N).\]

On the other hand, for \(N \geq 5\),
\begin{equation}
\int_{\Sigma} U_{\epsilon,0}^2 = c_N^2 \int_{\Sigma} \left( \frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{N-2} = c_N^2 \epsilon^2 \int_{\mathbb{R}^N} \frac{1}{(1 + |y|^2)^{N-2}} dy
\end{equation}

Thus for \(N \geq 5\)
\[c(L) \leq \frac{\int_{\Sigma} (|\nabla U_{\epsilon,0}|^2 + L^2 U_{\epsilon,0}^2)}{\left( \int_{\Sigma} U_{\epsilon,0}^{p+1} \right)^{\frac{N-2}{N}}} = \frac{A_0 - B_0 \epsilon^{N-2} + L^2 C_0 \epsilon^2 + O(\epsilon^N)}{(D_0 + O(\epsilon^N))^{\frac{N-2}{N}}} < \frac{A_0}{D_0^{\frac{N-2}{N}}} = S_{\frac{1}{2}}\]

if \(L\) is small and \(\epsilon\) is small. Here
\begin{equation}
A_0 = \int_{\mathbb{R}^N} |\nabla U_{1,0}|^2, B_0 = c_N^2 \int_{\mathbb{R}^{N-1}} \frac{1}{(1 + |y'|^2)^{N-1}} dy', C_0 = \int_{\mathbb{R}^N} U_{1,0}^2, D_0 = \int_{\mathbb{R}^N} U_{1,0}^{\frac{2N}{N-2}}
\end{equation}

Applying Lemma (4.1), for \(L\) small, \(c(L)\) is attained (possibly by a trivial solution).

For \(N = 4\), we have
\begin{equation}
\int_{\Sigma} U_{\epsilon,0}^2 = c_N^2 \epsilon^2 \log \frac{1}{\epsilon} \int_{\mathbb{R}^{N-1}} \frac{1}{(1 + |y'|^2)^2} dy'
\end{equation}

Similar arguments as before show that \(c(L) < S_{\frac{1}{2}}\) for \(L\) small.

\[\Box\]

Remark: In fact, Corollary 4.2 is also true for \(N \geq 3\). Another proof is to use the inequality \(c(L) \leq c^*(L)\).

4.2. Critical exponent case II: \(L\) large. The main theorem in this section is the following

**Theorem 4.3.** For \(L\) large, \(c(L)\) is not attained.

We first assume that \(N \geq 5\). Later on we will show how one can modify the arguments to deal with the case of \(N = 4\).

We prove it by contradiction. Suppose that \(c(L)\) is attained by some \(u_L\) for a sequence of \(L = L_i \to +\infty\). Note that the computations of Corollary 4.2 show that
\begin{equation}
c(L) \leq S_{\frac{1}{2}}.
\end{equation}
We claim that $c(L)$ is attained for all $L$. In fact, let $L < L_1$ for some $L_1$. 
Then we have
\begin{equation}
(4.12) \quad c(L) \leq \frac{\int_{\Sigma} (|\nabla u_{L_1}|^2 + L^2 u_{L_1}^2)}{(\int_{\Sigma} u_{L_1}^{p+1})^{\frac{2}{p+1}}} < c(L_1) \leq S_{\frac{1}{2}}.
\end{equation}

By Lemma 4.1, $c(L)$ is attained.

By Steiner symmetrization, we may assume that $u_L$ is symmetric in $x'$ and increasing in $x_N$.

We rescale $\tilde{u}_L = L^{-\frac{2}{p+1}} u_L(L^{-1} y)$ and obtain that
\begin{equation}
(4.13) \quad c(L) = \frac{\int_{\Sigma_L} (|\nabla \tilde{u}_L|^2 + \tilde{u}_L^2)}{(\int_{\Sigma_L} \tilde{u}_L^{p+1})^{\frac{2}{p+1}}} \leq S_{\frac{1}{2}}
\end{equation}

where $\Sigma_L = \mathbb{R}^{N-1} \times (0, L)$. This implies that
\begin{equation}
(4.13) \quad \int_{\Sigma_L} (|\nabla \tilde{u}_L|^2 + \tilde{u}_L^2) \leq C, \quad \int_{\Sigma_L} \tilde{u}_L^{p+1} \leq C.
\end{equation}

We claim that $\tilde{u}_L$ must blow up. If not, by taking a subsequence of $L$ and extending $\tilde{u}_L$ to $\mathbb{R}^{N-1} \times (-L, L)$, we see that $v_L$ converges to a positive solution of
\[ \Delta v - v + v^{\frac{N+2}{N-2}} = 0, \ v \in H^1(\mathbb{R}^N) \]
which is impossible. In fact we have that
\[ \frac{1}{\tilde{u}_L(0)} \tilde{u}_L((\tilde{u}_L(0))^{-\frac{N-2}{2}}(y) \to U_{1, 0}(y) \]
in $C^2_{\text{loc}}(\mathbb{R}^N)$ as $L \to \infty$.

Let $\epsilon$ be such that
\begin{equation}
(4.14) \quad \tilde{u}_L(0) = \epsilon^{-\frac{N+2}{2}}
\end{equation}
and set
\begin{equation}
(4.15) \quad v_\epsilon(y) = \epsilon^{-\frac{N+2}{2}} \tilde{u}_L(\epsilon y).
\end{equation}

Then it is easy to see that $v_\epsilon$ satisfies
\begin{equation}
(4.16) \quad \Delta v_\epsilon - \epsilon^2 v_\epsilon + v_\epsilon^p = 0 \quad \text{in } \Sigma_{\epsilon}, \quad \frac{\partial v_\epsilon}{\partial v} = 0 \quad \text{on } \Sigma_{\epsilon}
\end{equation}
and that $v_\epsilon(y) \to U_{1, 0}(y)$ in $C^2_{\text{loc}}(\mathbb{R}^N)$.

We now require the following crucial estimate.

**Lemma 4.4.**
\begin{equation}
(4.17) \quad v_\epsilon(y) \leq \frac{C}{(1 + |y|^2)^{\frac{N+2}{2}}}.
\end{equation}
Proof. The derivation of this estimate follows exactly the proof of Theorem 2.1 of [14]. In fact, our situation is simpler as there is no need to straighten the boundary (see [14]).

Let $V_\epsilon$ be the unique solution of the following linear problem

$$
\Delta V_\epsilon - \epsilon^2 V_\epsilon + U_{1,0}^\frac{_{N+2}}{_{N-2}} = 0 \text{ in } \Sigma_\epsilon, \quad \frac{\partial V_\epsilon}{\partial v} = 0 \text{ on } \partial \Sigma_\epsilon.
$$

Now we decompose

$$
V_\epsilon = U_{1,0} - \varphi_\epsilon
$$

Then we have

$$
\begin{align*}
\Delta \varphi_\epsilon - \epsilon^2 \varphi_\epsilon + \epsilon^2 U_{1,0} &= 0 \text{ in } \Sigma_\epsilon, \\
\frac{\partial \varphi_\epsilon}{\partial v} &= \frac{\partial U_{1,0}}{\partial v} \text{ on } \partial \Sigma_\epsilon.
\end{align*}
$$

Let us set

$$
\varphi_\epsilon = \epsilon^2 \varphi_0(y) + \varphi_{\epsilon,1}
$$

where $\varphi_0$ is the unique solution of the following problem

$$
\Delta \varphi_0 + U_{1,0}(y) = 0 \text{ in } \mathbb{R}^N, \quad \varphi_0(y) = \varphi_0(|y|), \varphi_0 \to 0 \text{ as } |y| \to +\infty
$$

Note that since

$$
U_{1,0}(y) \leq \frac{C}{(1 + |y|)^{N-2}}
$$

where $N-2 > 2$, there exists a unique solution to (4.21).

Then $\varphi_{\epsilon,1}$ satisfies

$$
\begin{align*}
\Delta \varphi_{\epsilon,1} - \epsilon^2 \varphi_{\epsilon,1} - \epsilon^4 \varphi_0(y) &= 0 \text{ in } \Sigma_\epsilon, \\
\frac{\partial \varphi_{\epsilon,1}}{\partial v} &= \frac{\partial U_{1,0}}{\partial v}[U_{1,0} - (\epsilon^2)\varphi_0] \text{ on } \partial \Sigma_\epsilon.
\end{align*}
$$

We claim that

**Lemma 4.5.**

$$
|\varphi_{\epsilon,1}| \leq C \epsilon^{N-2} + o(\epsilon^2) \frac{1}{(1 + |y|)^{\frac{N+2}{2}}},
$$

Now we let

$$
v_\epsilon(y) = V_\epsilon(y) + \phi_\epsilon(y)
$$

Then $\phi_\epsilon$ satisfies

$$
\begin{align*}
\Delta \phi_\epsilon - \epsilon^2 \phi_\epsilon + (V_\epsilon + \phi_\epsilon)^{\frac{\frac{N+2}{2}}{N-2}} - U_{1,0}^{\frac{\frac{N+2}{2}}{N-2}} &= 0 \text{ in } \Sigma_\epsilon, \\
\frac{\partial \phi_\epsilon}{\partial v} &= 0 \text{ on } \partial \Sigma_\epsilon.
\end{align*}
$$

We also claim that
Lemma 4.6.

\[(4.26) \quad |\phi_\epsilon(y)| \leq C\epsilon^2 \frac{1}{(1 + |y|^2)^\frac{\alpha-2}{2}}.\]

Postponing the proofs of Lemma (4.5) and Lemma (4.6) to the appendix, we can conclude that we have a contradiction to (4.11) by establishing the following:

\[(4.27) \quad c(L) > S\frac{1}{\epsilon}.\]

First we note that

\[(4.28) \quad c(L) = \frac{\int_{\Sigma_L} (|\nabla v_\epsilon|^2 + \epsilon^2 v_\epsilon^2)}{(\int_{\Sigma_L} v_\epsilon^{p+1})^{\frac{2}{p+1}}}.\]

Then we have

\[
\int_{\Sigma_L} (|\nabla v_\epsilon|^2 + \epsilon^2 v_\epsilon^2) = \int_{\Sigma_L} (|\nabla (V_\epsilon + \phi_\epsilon)|^2 + \epsilon^2 (V_\epsilon + \phi_\epsilon)^2) \\
= \int_{\Sigma_L} (|\nabla V_\epsilon|^2 + \epsilon^2 V_\epsilon^2) + 2 \int_{\Sigma_L} (\nabla V_\epsilon \nabla \phi_\epsilon + \epsilon^2 V_\epsilon \phi_\epsilon) \\
+ \int_{\Sigma_L} (|\nabla \phi_\epsilon|^2 + \epsilon^2 \phi_\epsilon^2) \\
= I_1 + 2I_2 + I_3
\]

where \(I_1, I_2\) and \(I_3\) are defined by the three terms at the last equality.

Quantity \(I_1\) can be computed as follows:

\[(4.29) \quad I_1 = \int_{\Sigma_L} U_{1,0}^{p+1} V_\epsilon = \int_{\Sigma_L} U_{1,0}^{p+1} (U_{1,0} - \phi_\epsilon) \\
= \int_{\mathbb{R}^N_+} U_{1,0}^{p+1} - \epsilon^2 \int_{\mathbb{R}^N_+} U_{1,0}^2 + o(\epsilon^2)
\]

where we have used

\[(4.30) \quad \int_{\mathbb{R}^N_+} U_{1,0}^{p+1} = \int_{\mathbb{R}^N_+} U_{1,0}^2 > 0.
\]

For the quantity \(I_2\), from the equation for \(V_\epsilon\), it follows that

\[(4.31) \quad I_2 = \int_{\Sigma_L} U_{1,0}^{p+1} \phi_\epsilon = O(\epsilon^2).\]
By Lemma 4.5, we have
\[ I_3 = \int_{\Sigma_{\varepsilon}} (|\nabla (\phi_\varepsilon)|^2 + \epsilon^2 (\phi_\varepsilon)^2) = o(\epsilon^2). \]

So
\[ \int_{\Sigma_{\varepsilon}} (|\nabla v_\varepsilon|^2 + \epsilon^2 v_\varepsilon^2) \geq \int_{\mathbb{R}^N} |\nabla U_{1,0}|^2 - \epsilon^2 \int_{\mathbb{R}^N} U_{1,0}^2 + 2I_2 + o(\epsilon^2). \]

Next, using Lemma 4.5 again, we obtain that
\[
\int_{\Sigma_{\varepsilon}} v_{\varepsilon}^{p+1} = \int_{\Sigma_{\varepsilon}} (V_\varepsilon + \phi_\varepsilon)^{p+1} \\
= \int_{\Sigma_{\varepsilon}} V_{\varepsilon}^{p+1} + (p+1) \int_{\Sigma_{\varepsilon}} V_{\varepsilon}^p \phi_\varepsilon + o(\epsilon^2) \\
= \int_{\Sigma_{\varepsilon}} V_{\varepsilon}^{p+1} + (p+1) \int_{\Sigma_{\varepsilon}} U_{1,0}^p \phi_\varepsilon + o(\epsilon^2) \\
(4.33) = \int_{\mathbb{R}^N} U_{1,0}^{p+1} - (p+1)\epsilon^2 \int_{\mathbb{R}^N} U_{1,0}^2 + I_2 + o(\epsilon^2).
\]

Combining (4.32) and (4.33), we obtain that
\[
(4.34) c(L) \geq \frac{\int_{\mathbb{R}^N} |\nabla U_{1,0}|^2 - \epsilon^2 \int_{\mathbb{R}^N} U_{1,0}^2 + I_2}{(\int_{\mathbb{R}^N} U_{1,0}^{p+1} - (p+1)\epsilon^2 \int_{\mathbb{R}^N} U_{1,0}^2 + (p+1)I_2 + o(\epsilon^2))^\frac{p}{p+1}} \\
\geq S_{\frac{1}{2}} + \epsilon^2 \int_{\mathbb{R}^N} U_{1,0}^2 + o(\epsilon^2) + O(|I_2|^2) > S_{\frac{1}{2}}
\]

which proves (4.27). \(\square\)

Finally, when \(N = 4\), we have to replace \(\varphi_0(y)\) be the following function
\[ \varphi_0(y) = \log \frac{1}{1 + |y|^2} + (\Delta)^{-1} \left( \frac{1}{1 + |y|^2} \right) \]
and \(\epsilon^2\) by \(\epsilon^2 \log \frac{1}{\epsilon}\). The rest of the proof is unchanged.

4.3. **Completion of proof of (3) of Theorem 1.1.** Let \(p = \frac{N+2}{N-2}\) and \(N \geq 4\). By Corollary 4.2, \(c(L)\) is attained for \(L\) small. Set
\[
(4.35) L_* = \sup \{ L | c(L) \text{ is attained for } l \in (0, L) \}.
\]

By Corollary 4.2 and Theorem 4.3, \(0 < L_* < +\infty\).

We claim that \(c(L)\) is also attained at \(L = L_*\). In fact, if not, let \(u_L, L < L_*\) be the minimizers of \(c(L)\). Then as \(L \to L_*\), \(u_L\) must blow up. But similar arguments as in Theorem 4.3 shows that this is impossible.
Now we show that for $L > L_*$, $c(L)$ is not attained. In fact, suppose $c(L)$ is attained for some $L_0 > L_*$ then certainly, $c(L_0) \leq S_{\frac{1}{2}}$. Let the minimizer of $c(L_0)$ be $u_{L_0}$. Then

$$\tag{4.36} c(L) < c(L_0) \leq S_{\frac{1}{2}}, \text{ for } L < L_0.$$ 

By Lemma 4.1, $c(L)$ is attained if $L < L_0$, which is a contradiction with the definition of $L_*$. 

Finally for $L$ small, we have $c(L) < c\left(\frac{L_*}{2}\right) \leq S_{\frac{1}{2}}$. By the same analysis as in Lemma 2.1 the minimizer is uniformly bounded. Hence for $L$ small $c(L)$ is archived by trivial solution. Now similar proofs as in Lemma 2.4 yield another constant $L_* < L_*$ such that for $L \leq L_*$, $c(L)$ is achieved by trivial solution and for $L \in (L_*, L_*], c(L)$ is attained by a nontrivial constant. 

Thus, (3) of Theorem 1.1 is proved. \hfill \Box

**Appendix A: Proof of Lemma 4.5 and Lemma 4.6.**

To prove Lemma 4.5 and 4.6, we introduce two weighted $L^\infty$ spaces. For $f$ a function in $\Sigma_\frac{L}{2}$, we define the following weighted $L^\infty$-norms 

$$\|f\|_* = \sup_{x \in \Sigma_\frac{L}{2}} \left(1 + |y|^2\right)^{\frac{N-2}{2}} |f(x)|$$

and

$$\|f\|_{**} = \sup_{x \in \Sigma_\frac{L}{2}} \left(1 + |y|^2\right)^{\frac{N-4}{2}} |f(x)|.$$

**Lemma A.** Let $f \in L^\infty(\Sigma_\frac{L}{2})$ be such that

$$\|f\|_{**} < +\infty$$

and $u$ satisfy

$$-\Delta u + \epsilon^2 u = f \text{ in } \Sigma_\frac{L}{2}, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Sigma_\frac{L}{2}.$$  

Then we have

$$\tag{4.37} |u(x)| \leq \int_{\Sigma_\frac{L}{2}} \frac{C}{|x-y|^{N-2}} |f(y)|dy,$$

where $C$ is independent of $L \geq L_0$. As a consequence, we have

$$\tag{4.38} \|u\|_* \leq C\|f\|_{**}.$$

To prove Lemma 4.5, we decompose $\varphi_{\epsilon,1}$ into two parts:

$$\varphi_{\epsilon,1} = \varphi_{\epsilon,1}^1 + \varphi_{\epsilon,1}^2.$$
where \( \varphi_{e,1} \) satisfies
\[
\Delta \varphi_{e,1}^1 - \epsilon^2 \varphi_{e,1} = 0 \text{ in } \Sigma_L, \quad \frac{\partial \varphi_{e,1}^1}{\partial \nu} = \frac{\partial}{\partial \nu} (U_1 - \epsilon^2 \varphi_0(y)) \text{ on } \partial \Sigma_L,
\]
and \( \varphi_{e,1}^2 \) satisfies
\[
\Delta \varphi_{e,1}^2 - \epsilon^2 \varphi_{e,1}^2 - \epsilon^2 \varphi_0(y) = 0 \text{ in } \Sigma_L, \quad \frac{\partial \varphi_{e,1}^2}{\partial \nu} = 0 \text{ on } \partial \Sigma_L.
\]

The first part can be estimated by asymptotic analysis while the second part follows from comparison principle.

To prove Lemma 4.6, we note that \( \phi_e \) satisfies
\[
\Delta \phi_e - \epsilon^2 \phi_e + p V_{e}^{p-1} \phi_e + E_e + N_e[\phi_e] = 0
\]
where
\[
E_e = V_{e}^p - U_{1,0}^p,
\]
\[
N_e[\phi_e] = (V_e + \phi_e)^p - V_{e}^p - p V_{e}^{p-1} \phi_e.
\]

We argue by contradiction. Let \( \tilde{\phi}_e = \frac{\phi_e}{\epsilon^2} \). We assume that
\[
(4.39) \quad \|\tilde{\phi}_e\|_* \to +\infty.
\]
Set
\[
(4.40) \quad \Phi_e = \frac{\tilde{\phi}_e}{\|\tilde{\phi}_e\|_*}.
\]

Then it is easy to see that \( \Phi_e \) satisfies
\[
\Delta \Phi_e - \epsilon^2 \Phi_e + p V_e^{p-1} \Phi_e + (\|\tilde{\phi}_e\|_*)^{-1} \epsilon^{-2} E_e + (\|\tilde{\phi}_e\|_*)^{-1} \epsilon^{-2} N_e = 0.
\]
As \( \epsilon \to 0 \), \( \Phi_e \to \Phi_0 \) where \( \Phi_0 \) satisfies
\[
\Delta \Phi_0 + p U_{1,0}^{p-1} \phi_0 = 0 \text{ in } \mathbb{R}^N.
\]

It is well-known (see [26]) that \( \Phi_0 = a_0 \frac{\partial U_{1,0}}{\partial \lambda} \big|_{\lambda=1} + \sum_{j=1}^{N-1} a_j \frac{\partial U_{1,0}}{\partial y_j} \) for some constants \( a_j, j = 0, 1, \ldots, N-1 \).

Now since both \( v_e \) and \( V_e \) are symmetric in \( x' \), we see that \( \frac{\partial}{\partial y_j} \Phi_e(0) = 0, j = 1, \ldots, N-1 \). We also have that \( \Phi_e(0) = \frac{v_e(0) - V_e(0)}{\epsilon\|\phi_e\|_*} = o(1) \) and hence \( \Phi_0(0) = 0 \).

This together with \( \frac{\partial}{\partial y_j} \phi_0(0) = 0 \) will force \( a_j = 0, j = 0, 1, \ldots, N-1 \) and hence \( \Phi_0 = 0 \).

On the other hand, from the equation for \( \Phi_e \), we have that
\[
\|V_e^{p-1} \Phi_e\|_{**} = o(1), \quad \|\epsilon^{-2} E_e\|_{**} = O(1),
\]
and by Lemma A, we then arrive at
\[
\|\Phi_e\|_* = o(1)
\]
A contradiction to (4.40)!
So Lemma 4.5 is proved.

It remains to prove Lemma A. By a scaling, we may assume that \( \epsilon = 1 \).

Then

\[
u(x) = \int_{\Sigma_L} G_L(x, y) f(y) dy
\]

where \( G_L(x, y) \) is the Green’s function

\[
\Delta G_L - G_L + \delta \xi = 0 \text{ in } \Sigma_L, \quad \frac{\partial G_L}{\partial \nu} = 0 \quad \text{on } \partial \Sigma_L.
\]

We have to show that

\[
(4.41) \quad G_L(x, y) \leq \frac{C}{|x - y|^{N-2}}
\]

where \( C \) is independent of \( L \geq L_0 \), But (4.41) follows from standard potential estimates. See [27]. Note also that \( |f(y)| \leq \frac{C}{(1 + |y|)^{N-2}} < \frac{C}{(1 + |y|)^{N-2}} \) so the integral

\[
\int_{\Sigma} \frac{|f(y)|}{|x - y|^{N-2}} \leq C \|f\|_{L^p} \leq \frac{1}{(1 + |y|)^{N-2}}.
\]

**Appendix B: Asymptotic behavior when \( L \to L^* \)**

In this appendix, we study the asymptotic behavior of the least energy solution when \( L \to L^* \). Let \( L = L_* \) and \( u_0 \) be the unique radial solution of (2.7). It is clear that when \( L \to L_* \), \( u_L \to u_0 \) uniformly. In the following, we shall derive the next two order terms in the expansion of \( u_L \).

First, we consider the following linear problem

\[
(4.42) \quad \Delta \phi - L_*^2 \phi + p u_0^{p-1} \phi = 0 \text{ in } \Sigma, \quad \phi \in H^1(\Sigma), \quad \frac{\partial \phi}{\partial \nu} = 0 \text{ on } \Sigma
\]

Then by separation of variables we have

\[
(4.43) \quad \phi = \sum_{j=1}^{N-1} \frac{\partial w_0}{\partial y_j} + c_N \Phi_0
\]

where

\[
(4.44) \quad \Phi_0 = \phi_0(|y'|) \cos(\pi y)
\]

and \( \phi_0 \) is the principal eigenfunction of \( \Delta_{y'} - L_*^2 + p u_0^{p-1} \). (We choose \( \Phi_0 \) so that \( \int_\Sigma \Phi_0^2 = 1 \).)

As a consequence of the Fredholm Alternative, there exists a unique solution to the following problem

\[
(4.45) \quad \Delta \phi - L_*^2 \phi + p u_0^{p-1} \phi = f \text{ in } \Sigma, \quad \phi \in H^1(\Sigma), \quad \frac{\partial \phi}{\partial \nu} = 0 \text{ on } \Sigma
\]

with

\[
(4.46) \quad \int_{\Sigma} \phi \frac{\partial w_0}{\partial y_j} = \int_{\Sigma} \phi \Phi_0 = \int_{\Sigma} f \frac{\partial w_0}{\partial y_j} = \int_{\Sigma} f \Phi_0 = 0
\]
such that
\[(4.47) \quad \| \phi \|_{H^2(\Sigma)} \leq C \| f \|_{L^2(\Sigma)}\]
Let us denote \( L_0 := \Delta - L_z^2 + p w_0^{p-1} \) and \( \mathcal{K}_0 = \text{span} \{ \partial w_0 / \partial y_j, \phi_0 \} \). Now set
\[(4.48) \quad \delta = \int_{\Sigma} (u_L - w_0) \phi_0 \]
We now claim that
\[(4.49) \quad u_L = w_0 + \delta \phi_0 + \delta^2 \phi_1 + \delta^3 \phi_2 \]
where \( \| \phi_2 \|_{H^2} \leq C \) and \( \phi_1 \) satisfies
\[(4.50) \quad L_0 \phi_1 + \frac{L_z^2 - L^2}{\delta^2} w_0 + \frac{p(p-1)}{2} w^{p-2} \phi_0^2 = 0 \text{ in } \Sigma, \phi_1 \perp \mathcal{K}_0, \frac{\partial \phi_1}{\partial \nu} = 0 \text{ on } \Sigma.\]
We prove it by expansion. Let \( u_L = w_0 + \delta \phi_0 + \phi_{1,L} \). Then \( \phi_{1,L} \perp \mathcal{K}_0 \) and \( \phi_{1,L} = o(1) \). Since \( \Delta (w_0 + \delta \phi_0) - L^2 (w_0 + \delta \phi_0) + (w_0 + \delta \phi_0)^p = (L_z^2 - L^2) w_0 + \delta^2 \frac{p(p-1)}{2} w^{p-2} \phi_0^2 + \mathcal{O}(|\delta|^3 + |L_z^2 - L^2||\delta|) \). We conclude that \( \phi_{1,L} = \mathcal{O}(|L_z^2 - L^2| + \delta^2) \). Now we let \( \phi_1 \) be as defined in (4.50). Decomposing \( u \) as in (4.49), we see that \( \phi_2 \) satisfies
\[ L_0 [\phi_2] + \frac{L_z^2 - L^2}{\delta^2} \phi_0 + p(p-1) w^{p-2} \phi_0 \phi_1 + \frac{p(p-1)(p-2)}{6} w^{p-3} \phi_0^3 + \mathcal{O}(\delta) = 0 \]
and hence we have that
\[ \frac{L_z^2 - L^2}{\delta^2} \int_{\Sigma} \phi_0^2 + p(p-1) w^{p-2} \int_{\Sigma} \phi_0 \phi_1 + \frac{p(p-1)(p-2)}{6} \int_{\Sigma} w^{p-3} \phi_0^3 = 0 \]
which gives the desired precise formula for \( \delta \) in terms of \( L_z^2 - L^2 \).

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